

On Chiral Quantum Superspaces

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Abstract

We give a quantum deformation of the chiral Minkowski superspace in 4 dimensions embedded as the big cell into the chiral conformal superspace. Both deformations are realized as quantum homogeneous superspaces: we deform the ring of regular functions together with a coaction of the corresponding quantum supergroup.

1 Introduction

In his foundational work on supergeometry [22] Manin realized the Minkowski superspace as the big cell inside the flag supermanifold of $2|0$ and $2|1$ superspaces in the superspace of dimension $4|1$.

In his construction however, the actions of the Poincaré and the conformal supergroups on the super Minkowski and its compactification were left in the background and did not play a crucial role. Moreover there was no explicit construction of the coordinate rings associated with the super Minkowski

space and the conformal superspace together with their embedding into a suitable projective superspace. Such coordinate rings are necessary in order to construct a quantum deformation.

Our intention is to fill this gap, by bringing the supergroup action to the center of the stage so that we can give explicitly the coordinate rings of the super Minkowski and conformal superspaces together with their embeddings into projective superspace. This will be our starting point to build a quantum deformation of them. We shall concentrate our attention in realizing the chiral super Minkowski space as the big cell in the super Grassmannian variety of $2|0$ superspaces in $\mathbb{C}^{4|1}$ (the chiral conformal superspace). This is not precisely the same supervariety that Manin considers in his work; the Grassmannian is a simpler one, but it also has a physical meaning. Our choice is motivated because in some supersymmetric theories *chiral superfields* appear naturally. Chiral superfields, in our approach, are identified with elements of the coordinate superalgebra of the above mentioned Grassmannian. If one wants to formulate certain supersymmetric field theories in a noncommutative superspace one needs to have the notion of quantum chiral superfields. It is not obvious in other approaches how to construct a quantum chiral superalgebra without loosing other properties, as the action of the group, for example. In our construction the quantum chiral superfields appear naturally together with the supergroup action.

We plan to explore in a forthcoming paper Manin's construction in this new framework.

We shall not go into the details of the proofs of all of our statements, since an enlarged version of part of this work is available in Ref. [3]; nevertheless we shall make a constant effort to convey the key ideas and steps of our constructions.

This is the content of the present paper.

In section 2 we briefly outline few key facts of supergeometry, favouring intuition over rigorous definitions. Our main reference will be Ref. [2].

In section 3 we discuss the chiral conformal superspace as an homogeneous superspace identified with the super Grassmannian variety of $2|0$ superspaces in the complex vector superspace of dimension $4|1$. We also provide an explicit projective embedding of the super Grassmannian into a suitable projective superspace.

In Section 4 we give an equivalent approach via invariant theory to the theory discussed in Section 3.

In Section 5 we introduce the complex super Minkowski space as the big cell in the chiral conformal superspace. We also provide an explicit description of the action of the super Poincaré group.

In Sections 6 and 7 we build a quantum deformation of the Minkowski superspace and its compactification together with a coaction of the quantum Poincaré and conformal supergroups.

Finally in Section 8 we discuss some relevant physical applications of the theory developed so far.

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2 Basic concepts in Supergeometry

Supergeometry is essentially \mathbb{Z}_2 -graded geometry: any geometrical object is given a \mathbb{Z}_2 -grading in some natural way and the morphisms are the maps respecting the geometric structure and the \mathbb{Z}_2 -grading.

For instance, a super vector space V is a vector space where we establish a \mathbb{Z}_2 -grading by giving a splitting $V_0 \oplus V_1$. The elements in V_0 are called *even* and the elements in V_1 are called *odd*. Hence we have a function p called the *parity* defined only on homogeneous elements. A *superalgebra* A is a super vector space with multiplication preserving parity. The *reduced superalgebra* associated with A is $A_r := A/I_{\text{odd}}$, where I_{odd} is the ideal generated by the odd nilpotents. Notice that the reduced superalgebra A_r may have even nilpotents, thus making the terminology a bit awkward.

A superalgebra A is *commutative* if

$$xy = (-1)^{p(x)p(y)}yx$$

for all x, y homogeneous elements in A . From now on we assume all superalgebras are to be commutative unless otherwise specified and their category is

denoted with (salg). We also need to introduce the notion of *affine superalgebra*. This is a finitely generated superalgebra such that A_r has no nilpotents. In ordinary algebraic geometry such A_r 's are associated bijectively to affine algebraic varieties, as we are going to see.

The most interesting objects in supergeometry are the *algebraic supervarieties* and the *differentiable supermanifolds*. Both these concepts are encompassed by the idea of *superspace*.

Definition 2.1. We define *superspace* the pair $S = (|S|, \mathcal{O}_S)$ where $|S|$ is a topological space and \mathcal{O}_S is a sheaf of superalgebras such that the stalk at a point $x \in |S|$ denoted by $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$.

A *morphism* $\phi : S \rightarrow T$ of superspaces is given by $\phi = (|\phi|, \phi^\#)$, where $|\phi| : |S| \rightarrow |T|$ is a map of topological spaces and $\phi^\# : \mathcal{O}_T \rightarrow \phi_* \mathcal{O}_S$ is a sheaf morphism such that $\phi_x^\#(\mathbf{m}_{|\phi|(x)}) = \mathbf{m}_x$ where $\mathbf{m}_{|\phi|(x)}$ and \mathbf{m}_x are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively.

Let us see an important example.

Example 2.2. The superspace $\mathbb{R}^{p|q}$ is the topological space \mathbb{R}^p endowed with the following sheaf of superalgebras. For any $U \subset_{\text{open}} \mathbb{R}^p$

$$\mathcal{O}_{\mathbb{R}^{p|q}}(U) = C^\infty(\mathbb{R}^p)(U) \otimes \mathbb{R}[\xi^1, \dots, \xi^q],$$

where $\mathbb{R}[\xi_1, \dots, \xi_q]$ is the exterior algebra (or *Grassmann algebra*) generated by the q variables ξ_1, \dots, ξ_q .

Definition 2.3. A *supermanifold* of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to the superspace $\mathbb{R}^{p|q}$, i. e. for all $x \in |M|$ there exist an open set $V_x \subset |M|$ and $U \subset \mathbb{R}^{p|q}$ such that:

$$\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{R}^{p|q}}|_U.$$

We shall now concentrate on the study of algebraic supervarieties, since our purpose is to obtain quantum deformations and for this reason the algebraic approach is to be preferred.

There are two equivalent and quite different approaches to both, algebraic supervarieties and differentiable supermanifolds: the sheaf theoretic and the functor of points categorical approach. In the first of these approaches an

algebraic supervariety (resp. a supermanifold) is to be understood as a superspace, that is a pair consisting of a topological space and a sheaf of superalgebras. In the special cases of an affine algebraic supervariety (resp. a differentiable supermanifold), the superalgebra of global sections of the sheaf allows us to reconstruct the whole sheaf and the underlying topological space (see [2] ch. 4 and 10). Consequently an affine supervariety (resp. a differentiable supermanifold) can be effectively identified with a commutative superalgebra.

This is the super counterpart to the well known result of ordinary complex algebraic geometry: affine varieties are in one-to-one correspondence with their coordinate rings, in other words, we associate the zeros of a set of polynomials into some affine space to the ideal generated by such polynomials. For example we associate to the complex sphere in \mathbb{C}^3 , the coordinate ring $\mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

We also say that there is an equivalence of categories between the category of affine supervarieties and the category of affine superalgebras. Besides the above mentioned correspondence, this amounts to the fact that morphisms of affine varieties correspond to morphisms of the correspondent coordinate rings.

We can take the same point of view in supergeometry and give the following definition.

Definition 2.4. Let $\mathcal{O}(X)$ be an affine superalgebra. We define *affine supervariety* X associated with $\mathcal{O}(X)$ the superspace $(|X|, \mathcal{O}_X)$, where $|X|$ is the topological space of an ordinary affine variety, while \mathcal{O}_X is the (unique) sheaf of superalgebras, whose global sections coincide with $\mathcal{O}(X)$, and there exists an open cover U_i of $|X|$ such that

$$\mathcal{O}_X(U_i) = \mathcal{O}(X)_{f_i} = \left\{ \frac{g}{f_i} \mid g \in \mathcal{O}(X) \right\}$$

for suitable $f_i \in \mathcal{O}(X)_0$. (for more details see [8] ch. II and [2] ch. 10).

A *morphism* of affine supervarieties is a morphism of the underlying superspaces, though one readily see it corresponds (contravariantly) to a morphism of the corresponding coordinate superalgebras:

$$\{\text{morphisms } X \longrightarrow Y\} \quad \longleftrightarrow \quad \{\text{morphisms } \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)\}$$

We define *algebraic supervariety* a superspace which is locally isomorphic to an affine supervariety. \square

Example 2.5.

1. *The affine superspace.* We define the *polynomial superalgebra* as:

$$\mathbb{C}[x^1, \dots, x^p, \theta^1, \dots, \theta^q] := \mathbb{C}[x^1, \dots, x^p] \otimes \Lambda(\theta^1, \dots, \theta^q).$$

We want to interpret this superalgebra as the coordinate superring of a supervariety that we call the *affine superspace* of superdimension $p|q$, and we shall denote with the symbol $\mathbb{C}^{p|q}$ or $\mathbf{A}^{m|n}$. The underlying topological space is \mathbf{A}^m , that is \mathbb{C}^m with the Zariski topology, while the sheaf is:

$$\mathcal{O}_{\mathbf{A}^{m|n}}(U) := \mathcal{O}_{\mathbf{A}^m}(U) \otimes \Lambda(\theta^1 \dots \theta^n).$$

2. *The supersphere.* The superalgebra $\mathbb{C}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 + \eta_1 x + \eta_2 x_2 + \eta_3 x_3 - 1)$ is the superalgebra of the global sections of an affine supervariety whose underlying topological space is the unitary sphere in \mathbf{A}^3 .

The first important example of a supervariety which is not affine is given by the *projective superspace*.

Example 2.6.

1. *Projective superspace.* Consider the \mathbb{Z} -graded superalgebra $S = \mathbb{C}[x_0 \dots x_m, \xi_1 \dots \xi_n]$. For each r , $0 \leq r \leq m$, we consider the graded superalgebra

$$S[r] = \mathbb{C}[x_0, \dots, x_m, \xi_1, \dots, \xi_n][x_r^{-1}], \quad \deg(x_r^{-1}) = -1.$$

The subalgebra $S[r]^0 \subset S[r]$ of \mathbb{Z} -degree 0 is

$$S[r]^0 \approx \mathbb{C}[u_0, \dots, \hat{u}_r, \dots, u_m, \eta_1, \dots, \eta_n], \quad u_s = \frac{x_s}{x_r}, \quad \eta_\alpha = \frac{\xi_\alpha}{x_r}, \quad (1)$$

(the ‘ $\hat{}$ ’ means that this generator is omitted). This is an affine superalgebra and it corresponds to an affine superspace, (see 2.5) whose topological space we denote with $|U_r|$ and the corresponding sheaf with \mathcal{O}_{U_r} . Notice that the topological spaces $|U_r|$ form an affine open cover of $|\mathbf{P}^m|$, the ordinary projective space of dimension m .

A direct calculations shows that:

$$\mathcal{O}_{U_r}|_{|U_r| \cap |U_s|} = \mathcal{O}_{U_s}|_{|U_r| \cap |U_s|},$$

so we conclude that there exists a unique sheaf on the topological space $|\mathbf{P}^m|$, that we denote as $\mathcal{O}_{\mathbf{P}^{m|n}}$, whose restriction to $|U_i|$ is \mathcal{O}_{U_i} . Hence we

have defined a supervariety that we denote with $\mathbf{P}^{m|n}$ and call the *projective superspace* of dimension $m|n$.

2. Projective supervarieties.

Let $I \subset S = \mathbb{C}[x_1 \dots x_m, \xi_1 \dots \xi_n]$ be a homogeneous ideal; then S/I is also a graded superalgebra and we can repeat the same construction as above. First of all, we notice that the reduced algebra $(S/I)_r$ corresponds to an ordinary projective variety, whose topological space we denote with $|X|$, embedded into a projective superspace $|X| \subset |\mathbf{P}^m|$. Consider the superalgebra of \mathbb{Z} -degree zero elements in $(S/I)[x_i^{-1}]$ (this is called *projective localization*):

$$\left(\frac{\mathbb{C}[x_0, \dots, x_m, \xi_1 \dots \xi_n]}{I} [x_i^{-1}] \right)_0 \cong \frac{\mathbb{C}[u_0, \dots, \hat{u}_i, \dots, u_m, \eta_1 \dots \eta_n]}{I_{\text{loc}}},$$

where I_{loc} are the even elements of \mathbb{Z} -degree zero in $I[x_i^{-1}]$.

Again this affine superalgebra defines an affine supervariety with topological space $|V_i| \subset |U_i| \subset |\mathbf{P}^m|$ and sheaf \mathcal{O}_{V_i} . One can check that the supersheaves \mathcal{O}_{V_i} are such that $\mathcal{O}_{V_i}|_{|V_i| \cap |V_j|} = \mathcal{O}_{V_j}|_{|V_i| \cap |V_j|}$, so they glue to give a sheaf on $|X|$. Hence as before there exists a supervariety corresponding to the homogeneous superring S/I . This supervariety comes equipped with a projective embedding, encoded by the morphism of graded superalgebra $S \rightarrow S/I$, hence $(|X|, \mathcal{O}_X)$ is called a *projective supervariety*. \square

It is very important to remark that, contrary to the affine case, there is no coordinate superring associated intrinsically to a projective supervariety, but there is a coordinate superring associated with the projective supervariety and its projective embedding. In other words we can have the same projective variety admitting non isomorphic coordinate superrings with respect to two different projective embeddings.

We now want to introduce the functor of points approach to the theory of supervarieties.

Classically we can examine the points of a variety over different fields and rings. For example we can look at the rational points of the complex sphere described above. They are in one to one correspondence with the morphisms: $\mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 - 1) \rightarrow \mathbb{Q}$. In fact each such morphism is specified by the knowledge of the images of the generators. The idea behind the functor of points is to extend this and consider *all* morphisms from the coordinate ring of the affine supervariety to *all* superalgebras at once.

Definition 2.7. Let $\mathcal{A} \in (\text{salg})$, the category of commutative superalgebras. We define the \mathcal{A} -points of an affine supervariety X as the (superalgebra) morphisms $\text{Hom}(\mathcal{O}(X), \mathcal{A})$. We define the functor of points of X as:

$$h_X : (\text{salg}) \longrightarrow (\text{sets}), \quad h_X(\mathcal{A}) = \text{Hom}(\mathcal{O}(X), \mathcal{A}).$$

In other words $h_X(\mathcal{A})$ are the \mathcal{A} -points of X , for all commutative superalgebras \mathcal{A} .

Example 2.8. If \mathcal{A} is a generic (commutative) superalgebra, an \mathcal{A} -point of $\mathbb{C}^{p|q}$ (see Example 2.5) is given by a morphism $\mathbb{C}[x^1, \dots, x^p, \theta^1, \dots, \theta^q] \longrightarrow \mathcal{A}$, which is determined once we know the image of the generators

$$(x^1, \dots, x^p, \theta^1, \dots, \theta^q) \longrightarrow (a^1, \dots, a^p, \alpha^1, \dots, \alpha^q),$$

with $a^i \in \mathcal{A}_0$ and $\alpha^j \in \mathcal{A}_1$. Notice that the \mathbb{C} -points of $\mathbb{C}^{p|q}$ are given by $(k_1 \dots k_p, 0 \dots 0)$ and coincide with the points of the affine space \mathbb{C}^p . In this example it is clear that the knowledge of the points over a field is by no means sufficient to describe the supergeometric object.

Remark 2.9. It is important at this point to notice that just giving a functor from (salg) to (sets) , does not guarantee that it is the functor of points of a supervariety. A set of conditions to establish this is given in [2] ch. 10.

The functor of points for projective supervarieties is more complicated and we are unable to give a complete discussion here. it would be too long to give a general discussion here. We shall nevertheless discuss the functor of points of the projective space and superspace.

Example 2.10. Let us consider the functor: $h : (\text{alg}) \longrightarrow (\text{sets})$, where $h(\mathcal{A})$ are the projective \mathcal{A} -modules of rank one in \mathcal{A}^n .

Equivalently $h(\mathcal{A})$ consists of the pairs (L, ϕ) , where L is a projective \mathcal{A} -module of rank one, and ϕ is a surjective morphisms $\phi : \mathcal{A}^{n+1} \longrightarrow L$. These pairs are taken modulo the equivalence relation

$$(L, \phi) \approx (L', \phi') \quad \Leftrightarrow \quad L \stackrel{a}{\approx} L', \quad \phi' = a \circ \phi,$$

If $\mathcal{A} = \mathbb{C}$, then projective modules are free and a morphism

$$\phi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$$

is specified by a n -tuple, (a^1, \dots, a^{n+1}) , with $a^i \in \mathbb{C}$, not all of the $a^i = 0$. The equivalence relation becomes

$$(a^1, \dots, a^{n+1}) \sim (b^1, \dots, b^{n+1}) \iff (a^1, \dots, a^{n+1}) = \lambda(b^1, \dots, b^{n+1}),$$

with $\lambda \in \mathbb{C}^\times$ understood as an automorphism of \mathbb{C} . It is clear then that $h(\mathbb{C})$ consists of all the lines through the origin in the vector space \mathbb{C}^{n+1} , thus recovering the usual definition of complex projective space.

If \mathcal{A} is local, projective modules are free over local rings. We then have a situation similar to the field setting: equivalence classes are lines in the \mathcal{A} -module \mathcal{A}^{n+1} .

Using the Representability Theorem (see [2]) one can show that the functor h is the functor of points of a variety that we call the projective space and whose geometric points coincide with the projective space \mathbf{P}^n over the field k as we usually understand it. \square

This example can be easily generalized to the supercontext: we consider the functor $h_{\mathbf{P}^{m|n}} : (\text{salg}) \longrightarrow (\text{sets})$, where $h_{\mathbf{P}^{m|n}}(\mathcal{A})$ is defined as the set the projective \mathcal{A} -modules of rank one in $\mathcal{A}^{m|n} := \mathcal{A} \otimes \mathbb{C}^{m|n}$. This is the functor of points of the *projective superspace* described in Example 2.6.

The next question that we want to tackle is how we can define an embedding of a (super)variety into the projective (super)space using the functor of points notation.

Let X be a projective supervariety and $\Phi : X \longrightarrow \mathbf{P}^{m|n}$ be an injective morphism. As we discussed in Example 2.6 this embedding is encoded by a surjective morphism:

$$\mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n] \longrightarrow \mathbb{C}[x_1, \dots, x_m, \xi_1, \dots, \xi_n] / (f_1, \dots, f_r)$$

In the notation of the functor of points, Φ is a *natural transformation* between the two functors h_X and $h_{\mathbf{P}^{m|n}}$, given by

$$\Phi_{\mathcal{A}} : h_X(\mathcal{A}) \longrightarrow h_{\mathbf{P}^{m|n}}(\mathcal{A})$$

with $\Phi_{\mathcal{A}}$ injective.

If \mathcal{A} is a local superalgebra, then an \mathcal{A} -point $(a_1, \dots, a_m, \alpha_1, \dots, \alpha_n) \in h_{\mathbf{P}^{m|n}}(\mathcal{A})$ is in $\phi_{\mathcal{A}}(h_X(\mathcal{A}))$ if and only if it satisfies the homogeneous polynomial relations

$$\begin{aligned} f_1(a_1 \dots a_m, \alpha_1 \dots, \alpha_n) &= 0, \\ &\vdots \\ f_r(a_1 \dots a_m, \alpha_1 \dots, \alpha_n) &= 0. \end{aligned}$$

(See [3] for more details).

In summary, to determine the coordinate superalgebra of a projective supervariety with respect to a certain projective embedding, we need to check the relations satisfied by the coordinates *just on local superalgebras*. This will be our starting point when we shall determine the coordinate superalgebra of the Grassmannian supervariety with respect to its Plücker embedding.

3 The chiral conformal superspace

We are interested in the super Grassmannian of $(2|0)$ -planes inside the superspace $\mathbb{C}^{4|1}$, that we denote with Gr . This will be our chiral conformal superspace once we establish an action of the conformal supergroup on it.

Gr is defined via its functor of points. For a generic superalgebra \mathcal{A} , the \mathcal{A} -points of Gr consist of the projective modules of rank $2|0$ in $\mathcal{A}^{4|1} := \mathcal{A} \otimes \mathbb{C}^{4|1}$. It is not immediately clear that this is the functor of points of a supervariety, however a fully detailed proof of this fact is available in [3], Appendix A. Another important issue is the fact that once a supervariety is given, its functor of points is completely determined just by looking at the *local* superalgebras, and similarly the natural transformations are determined if we know them for local superalgebras. This is a well known fact that can be found for example in Ref. [16], Appendix A.

On a local superalgebra \mathcal{A} , $h_{\text{Gr}}(\mathcal{A})$ consists of free submodules of rank $2|0$ in $\mathcal{A}^{4|1}$ (on local superalgebras, projective modules are free). One such module can be specified by a couple of independent even vectors, a and b , which in the canonical basis $\{e_1, e_2, e_3, e_4, \mathcal{E}_5\}$ are given by two column vectors that span the subspace

$$\pi = \langle a, b \rangle = \left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \alpha_5 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \beta_5 \end{pmatrix} \right\rangle, \quad (2)$$

with $a_i, b_i \in \mathcal{A}_0$ and $\alpha_5, \beta_5 \in \mathcal{A}_1$. Let

$$h_{\mathrm{GL}(4|1)}(\mathcal{A}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \rho_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & \rho_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & \rho_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & \rho_{45} \\ \delta_{51} & \delta_{52} & \delta_{53} & \delta_{54} & d_{55} \end{pmatrix} \right\}, \quad (3)$$

define the functor of points of the supergroup $\mathrm{GL}(4|1)$, where $c_{ij}, d_{55} \in \mathcal{A}_0$ and $\rho_{i5}, \delta_{5i} \in \mathcal{A}_1$. We can describe the action of the supergroup $\mathrm{GL}(4|1)$ over Gr as a natural transformation of the functors (for \mathcal{A} local),

$$\begin{aligned} h_{\mathrm{GL}(4|1)}(\mathcal{A}) \times h_{\mathrm{Gr}}(\mathcal{A}) &\longrightarrow h_{\mathrm{Gr}}(\mathcal{A}) \\ g, \langle a, b \rangle &\longmapsto \langle g \cdot a, g \cdot b \rangle. \end{aligned}$$

Let $\pi_0 = \langle e_1, e_2 \rangle \in h_{\mathrm{Gr}}(\mathcal{A})$. The stabilizer of this point in $\mathrm{GL}(4|1)$ is the upper parabolic super subgroup P_u , whose functor of points is

$$h_{P_u}(\mathcal{A}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \rho_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & \rho_{25} \\ 0 & 0 & c_{33} & c_{34} & \rho_{35} \\ 0 & 0 & c_{43} & c_{44} & \rho_{45} \\ 0 & 0 & \delta_{53} & \delta_{54} & d_{55} \end{pmatrix} \right\} \subset h_{\mathrm{GL}(4|1)}(\mathcal{A}).$$

Then, the Grassmannian is identified with the quotient

$$h_{\mathrm{Gr}}(\mathcal{A}) = h_{\mathrm{GL}(4|1)}(\mathcal{A}) / h_{P_u}(\mathcal{A}).$$

We want now to work out the expression for the *Plücker embedding*. It is important to stress that, contrary to what happens in the classical setting, in the super context we have that a generic Grassmannian supervariety does not admit a projective embedding. However for this particular Grassmannian such embedding exists, as we are going to show presently.

We want to give a natural transformation among the functors

$$p : h_{\mathrm{Gr}} \rightarrow h_{\mathbf{P}(E)},$$

where E is the super vector space $E = \wedge^2 \mathbb{C}^{4|1} \approx \mathbb{C}^{7|4}$. Given the canonical basis for $\mathbb{C}^{4|1}$ we construct a basis for E

$$\begin{aligned} e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4, \mathcal{E}_5 \wedge \mathcal{E}_5, & \quad (\text{even}) \\ e_1 \wedge \mathcal{E}_5, e_2 \wedge \mathcal{E}_5, e_3 \wedge \mathcal{E}_5, e_4 \wedge \mathcal{E}_5, & \quad (\text{odd}) \end{aligned} \quad (4)$$

As in the super vector space case, if L is a \mathcal{A} -module, for $\mathcal{A} \in (\text{salg})$, we can construct $\wedge^2 L$

$$\wedge^2 L = L \otimes L / \langle u \otimes v + (-1)^{|u||v|} v \otimes u \rangle, \quad u, v \in L.$$

If $L \in h_{\text{Gr}}(\mathcal{A})$, then $\wedge^2 L \subset \wedge^2 \mathcal{A}^{4|1}$. It is clear that if L is a projective \mathcal{A} -module of rank $2|0$, then $\wedge^2 L$ is a projective \mathcal{A} -module of rank $1|0$. In other words it is an element of $h_{\mathbf{P}(E)}(\mathcal{A})$, for $E = \wedge^2 \mathbb{C}^{4|1}$. Hence we have defined a natural transformation:

$$\begin{array}{ccc} h_{\text{Gr}}(\mathcal{A}) & \xrightarrow{p} & h_{\mathbf{P}(E)}(\mathcal{A}) \\ L & \longrightarrow & \wedge^2 L. \end{array}$$

Once we have the natural transformation defined, we can again restrict ourselves to work only on local algebras.

Let a, b be two even independent vectors in $\mathcal{A}^{4|1}$. For any superalgebra \mathcal{A} , they generate a free submodule of $\mathcal{A}^{4|1}$ of rank $2|0$. The natural transformation described above is as follows.

$$\begin{array}{ccc} h_{\text{Gr}}(\mathcal{A}) & \xrightarrow{p_{\mathcal{A}}} & h_{\mathbf{P}(E)}(\mathcal{A}) \\ \langle a, b \rangle_{\mathcal{A}} & \longrightarrow & \langle a \wedge b \rangle. \end{array}$$

The map $p_{\mathcal{A}}$ is clearly injective. The image $p_{\mathcal{A}}(h_{\text{Gr}}(\mathcal{A}))$ is the subset of even elements in $h_{\mathbf{P}(E)}(\mathcal{A})$ decomposable in terms of two even vectors of $\mathcal{A}^{4|1}$. We are going to find the necessary and sufficient conditions for an even element $Q \in h_{\mathbf{P}(E)}(\mathcal{A})$ to be decomposable. Let

$$\begin{aligned} Q &= q + \lambda \wedge \mathcal{E}_5 + a_{55} \mathcal{E}_5 \wedge \mathcal{E}_5, \quad \text{with} \\ q &= q_{12} e_1 \wedge e_2 + \cdots + q_{34} e_3 \wedge e_4, \quad q_{ij} \in \mathcal{A}_0, \\ \lambda &= \lambda_1 e_1 + \cdots + \lambda_4 e_4, \quad \lambda_i \in \mathcal{A}_1. \end{aligned} \tag{5}$$

Q is decomposable if and only if

$$\begin{aligned} Q &= (r + \xi \mathcal{E}_5) \wedge (s + \theta \mathcal{E}_5) \quad \text{with} \\ r &= r_1 e_1 + \cdots + r_4 e_4, \quad s = s_1 e_1 + \cdots + s_4 e_4, \quad r_i, s_i \in \mathcal{A}_0 \quad \xi, \theta \in \mathcal{A}_1, \end{aligned}$$

which means

$$Q = r \wedge s + (\theta r - \xi s) \wedge \mathcal{E}_5 + \xi \theta \mathcal{E}_5 \wedge \mathcal{E}_5 \text{ equivalent to } q = r \wedge s, \quad \lambda = \theta r - \xi s, \quad a_{55} = \xi \theta.$$

These are equivalent to the following:

$$q \wedge q = 0, \quad q \wedge \lambda = 0, \quad \lambda \wedge \lambda = 2a_{55}q \quad \lambda a_{55} = 0.$$

Plugging (5) we obtain

$$\begin{aligned} q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} &= 0, & (\text{classical Plücker relation}) \\ q_{ij}\lambda_k - q_{ik}\lambda_j + q_{jk}\lambda_i &= 0, & 1 \leq i < j < k \leq 4 \\ \lambda_i\lambda_j &= a_{55}q_{ij} & 1 \leq i < j \leq 4 \\ \lambda_i a_{55} &= 0. \end{aligned} \tag{6}$$

These are the *super Plücker relations*. As we shall see in the next section the superalgebra

$$\mathcal{O}(\text{Gr}) = k[q_{ij}, \lambda_k, a_{55}]/\mathcal{I}_P, \tag{7}$$

is associated to the supervariety Gr in the Plücker embedding described above, where \mathcal{I}_P denotes the ideal of the super Plücker relations (6). In other words \mathcal{I}_P contains all the relations involving the coordinates q_{ij} , λ_k and a_{55} .

Remark 3.1. The superalgebra $\mathcal{O}(\text{Gr})$ is a sub superalgebra (though not a Hopf sub superalgebra) of $\mathcal{O}(\text{GL}(4|1))$. It is in fact the superalgebra generated by the corresponding minors, and the Plücker relations are all the relations satisfied by these minors in $\mathcal{O}(\text{GL}(4|1))$.

4 The super Grassmannian via invariant theory

In this section we propose an alternative and equivalent way to construct the super Grassmannian Gr as a complex supervariety and we give the coordinate superring associated to the super Grassmannian in the Plücker embedding, thus completing the discussion initiated in the previous section.

As we have seen in Section 2, the super Grassmannian can be equivalently understood as a pair consisting of the underlying topological space $G(2, 4)$, and a sheaf of superalgebras conveniently chosen that we shall describe presently.

We recall first what happens in the ordinary case. Let the set S be

$$S = \{(v, w) \in \mathbb{C}^4 \oplus \mathbb{C}^4 / \text{rank}(v, w) = 2\},$$

and consider the equivalence relation

$$(v, w) \sim (v', w') \Leftrightarrow \text{span}\{v, w\} = \text{span}\{v', w'\},$$

or equivalently

$$(v, w) \sim (v', w') \Leftrightarrow \exists g \in \text{GL}(2, \mathbb{C}) \text{ such that } (v', w') = (v, w)g.$$

Then we have that $G(2, 4) = S / \sim$.

We consider now the set of polynomials on S , $\text{Pol}(S)$, and the subset of such polynomials that is semi-invariant under the transformation of $\text{GL}(2, \mathbb{C})$, that is

$$f(v', w') = f(u, v)\lambda(g), \quad \lambda(g) \in \mathbb{C}, \quad f \in \text{Pol}(S).$$

This defines the homogeneous ring of $G(2, 4)$, which is generated by the six determinants [19].

$$y_{ij} = v_i w_j - v_j w_i, \quad \text{with } i < j \text{ and } \lambda = \det g.$$

These are not all independent, they satisfy the Plücker relation

$$y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24} = 0.$$

Let \mathcal{O} be the sheaf of polynomials on S , so for each open set in $\tilde{U} \subset S$, $\mathcal{O}(\tilde{U}) = \text{Pol}(\tilde{U})$ and \mathcal{O}^{inv} the subsheaf of \mathcal{O} corresponding to the semi-invariant polynomials.

Let $\pi : S \rightarrow G(2, 4)$ be the natural projection. It is clear that for $U \subset_{\text{open}} G(2, 4)$, then $\tilde{U} = \pi^{-1}(U) \subset S$ is also open in S . We can define the following sheaf over $G(2, 4)$:

$$\mathcal{O}(U) = \mathcal{O}^{\text{inv}}(\pi^{-1}(U)).$$

This is the structural sheaf of the projective variety $G(2, 4)$ with respect to the Plücker embedding.

Now we turn to the super setting and we want to define the sheaf of super-algebras generalizing the non super construction to the super Grassmannian. We define the superalgebra

$$\mathcal{F}(S) := \text{Pol}(S) \otimes \Lambda[\xi_1, \xi_2].$$

Let $(v, w) \in S$ and consider the (5×2) matrix

$$\begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_4 & w_4 \\ \xi_1 & \xi_2 \end{pmatrix}.$$

The group $\mathrm{GL}(2, \mathbb{C})$ acts on the right on these matrices

$$\begin{pmatrix} v' & w' \\ \xi'_1 & \xi'_2 \end{pmatrix} = \begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} \cdot g, \quad g \in \mathrm{GL}(2, \mathbb{C}).$$

We will write an element $f(v, w, \xi) \in \mathcal{F}(S)$ as

$$f(v, w, \xi) = \sum_{i,j=0,1} f_{ij}(v, w) \xi_1^i \xi_2^j.$$

We will refer to the elements of $\mathcal{F}(S)$ as ‘functions’, being this customary in the physics literature. We now consider the set of semi-invariant functions

$$f(v', w', \xi') = f(v, w, \xi) \lambda(g), \quad \lambda(g) \in \mathbb{C}, \quad f \in \mathcal{F}(S).$$

The following functions are semi-invariant:

$$y_{ij} = v_i w_j - v_j w_i, \quad \theta_i = v_i \xi_2 - w_i \xi_1, \quad a = \xi_1 \xi_2, \quad (8)$$

with $\lambda(g) = \det g$ but they are not all independent. They satisfy the *super Plücker relations* (6)

$$\begin{aligned} y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} &= 0, & (\text{standard Plücker relation}) \\ y_{ij}\theta_k - y_{ik}\theta_j + y_{jk}\theta_i &= 0 & 1 \leq i < j < k \leq 4 \\ \theta_i\theta_j &= ay_{ij} & 1 \leq i < j \leq 4 \\ \theta_i a &= 0 & 1 \leq i \leq 4 = 0. \end{aligned}$$

We want to show that the elements in (8) generate the ring of semi-invariants and that (6) are all the relations among these generators.

Proposition 4.1. *Let f be a homogeneous semi-invariant function, so*

$$f(v', w', \xi') = f(v, w, \xi) \lambda(g)$$

with

$$\begin{pmatrix} v' & w' \\ \xi'_1 & \xi'_2 \end{pmatrix} = \begin{pmatrix} v & w \\ \xi_1 & \xi_2 \end{pmatrix} \cdot g, \quad g \in \mathrm{GL}(2, \mathbb{C}).$$

Then in the decomposition

$$f(v, w, \xi) = f_0(v, w) + \sum_i f_i(v, w) \xi_i + f_{12}(v, w) \xi_1 \xi_2, \quad (9)$$

one has that $f_0(v, w)$ and $f_{12}(v, w)$ are standard (non-super) semi-invariants and

$$\sum_i f_i(v, w) \xi_i = \sum_i h_i(v, w) \theta_i,$$

with $h_i(v, w)$ also a standard semi-invariant.

Proof. Let us take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} v' & w' \\ \xi'_1 & \xi'_2 \end{pmatrix} = \begin{pmatrix} va + wc & vb + wd \\ \xi_1 a + \xi_2 c & \xi_1 b + \xi_2 d \end{pmatrix}.$$

Then we can see immediately that each term in (9) has to be a semi-invariant, so

$$f_0(v', w') = \lambda(g) f_0(v, w), \quad \sum_i f_i(v', w') \xi'_i = \lambda(g) \sum_i f_i(v, w) \xi_i,$$

$$f_{12}(v', w') \xi'_1 \xi'_2 = f_{12}(v, w) \xi_1 \xi_2.$$

We have that f_0 is an ordinary semi-invariant transforming with $\lambda(g)$, and since $\xi'_1 \xi'_2 = \xi_1 \xi_2 \det g$, $f_{12}(v, w)$ is a ordinary semi-invariant transforming with $\lambda(g) \det g^{-1}$. The odd terms θ^i are of the same form as the ordinary invariants y_{ij} , since the fact that ξ_i is odd plays no particular role here (recall that we are considering the action of an ordinary group, namely $\mathrm{GL}(2, \mathbb{C})$). So by the same argument we have in the ordinary case, there are no other odd invariants, besides those we have already found, that are linear in the odd variable ξ_1 and ξ_2 . Then

$$\sum_i f_i \xi_i = \sum_i h(v, w)_i \theta_i,$$

where $h(v, w)_i$ transforms with $\lambda(g) \det g^{-1}$. □

We now wish to give a result that describes completely the relations among the invariants.

Consider the polynomial superalgebra $\mathbb{C}[a_{ib}]$, $1 \leq i \leq 5$, $1 \leq b \leq 2$, with their parity defined as

$$p(a_{ij}) = p(i) + p(j), \quad \text{with} \quad p(k) = 0 \text{ if } 0 \leq k \leq 4 \text{ and } p(5) = 1.$$

On $\mathbb{C}[a_{ij}]$ there exists the following action of $\text{GL}(2, \mathbb{C})$:

$$\begin{aligned} \mathbb{C}[a_{ib}] \times \text{GL}(2, \mathbb{C}) &\longrightarrow \mathbb{C}[a_{ib}] \\ (a_{ia}, g^{-1}) &\longrightarrow \sum_k a_{ib} g_{ba}^{-1} \end{aligned}$$

We have just proven that the semi-invariants are generated by the polynomials

$$\begin{aligned} d_{ij} &= a_{i1}a_{j2} - a_{i2}a_{j1}, \quad 1 \leq i < j \leq 5, \\ d_{55} &= a_{51}a_{52}. \end{aligned}$$

We have the following proposition:

Proposition 4.2. *Let $\mathcal{O}(\text{Gr})$ be the subring of $\mathbb{C}[a_{ib}]$ generated by the determinants $d_{ij} = a_{i1}a_{j2} - a_{j1}a_{i2}$ and $d_{55} = a_{51}a_{52}$. Then $\mathcal{O}(\text{Gr}) \cong \mathbb{C}[a_{ib}]/I_P$, where I_P is the ideal of the super Plücker relations (6). In other words I_P contains all the possible relations satisfied by d_{ij} and d_{55} .*

Proof. It is easy to verify that d_{ij} and d_{55} satisfy all the above relations, the problem is to prove that these are the only relations.

The proof of this fact is the same as in the classical setting. Let us briefly sketch it. Let I_1, \dots, I_r be multiindices organized in a tableau. We say that a tableau is *superstandard* if it is strictly increasing along rows with the exception of the number 5 (that can be repeated) and weakly increasing along columns. A *standard monomial* in $\mathcal{O}(\text{Gr})$ is a monomial d_{I_1}, \dots, d_{I_r} where the indices I_1, \dots, I_r form a superstandard tableau. Using the super Plücker relation one can verify that any monomial in $\mathcal{O}(\text{Gr})$ can be written as a linear combination of standard ones. This can be done directly or using the same argument for the classical case (see Ref. [19] pg 110 for more details). The standard monomials are also linearly independent, hence they form a basis for $\mathcal{O}(\text{Gr})$ as \mathbb{C} -vector space. Again this is done with the same argument as in Ref. [19] pg 110. So given a relation in $\mathcal{O}(\text{Gr})$, once we write each term as a standard monomial we obtain that either the relation is identically zero (hence it is a relation in the Plücker ideal) or it gives a relation among the standard monomials, which gives a contradiction. \square

In the end we summarize the main results of Sections 3 and 4 with a corollary.

Corollary 4.3. *1. Let Gr be the Grassmannian of $2|0$ spaces in $\mathbb{C}^{4|1}$. Then $\text{Gr} \subset \mathbf{P}^{7|4}$, that is Gr is a projective supervariety. Such embedding is encoded by the superring $\mathcal{O}(\text{Gr})$ described above.*

2. $\mathcal{O}(\text{Gr})$ is isomorphic to the ring generated by the determinants d_{ij} , d_{55} .

5 The chiral Minkowski superspace

In this section we concentrate our attention to determine the big cell inside the Grassmannian supervariety that we have discussed in the previous sections. We shall identify such big cell with the chiral Minkowski superspace.

As in the ordinary setting, the super Grassmannian Gr admits an open cover in terms of affine superspaces: topologically the two covers are the same.

We want to describe the functor of points of the *big cell* U_{12} inside Gr . This is the open affine functor corresponding to the points in which the coordinate q_{12} is invertible.

First of all, we write an element of $h_{\text{GL}(4|1)}(\mathcal{A})$ in blocks as (see (3))

$$\begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix}.$$

Assuming that $\det C_1$ is invertible, we can bring this matrix, with a transformation of $h_{P_u}(\mathcal{A})$, to the form

$$\begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix} h_{P_u}(\mathcal{A}) = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 \\ A & \mathbb{1}_2 & 0 \\ \alpha & 0 & 1 \end{pmatrix} h_{P_u}(\mathcal{A}) \in h_{\text{GL}(4|1)}(\mathcal{A}) / h_{P_u}(\mathcal{A}) \quad (10)$$

Consider the subspace $\pi = \text{span}\{a, b\}$ in $h_{\text{Gr}}(\mathcal{A})$ for \mathcal{A} local. Recall that in Sec. 3 we made the identification: $h_{\text{Gr}}(\mathcal{A}) \cong h_{\text{GL}(4|1)}(\mathcal{A}) / h_{P_u}(\mathcal{A})$. Hence:

$$\pi = \text{span}\{a, b\} \approx \begin{pmatrix} C_1 & C_2 & \rho_1 \\ C_3 & C_4 & \rho_2 \\ \delta_1 & \delta_2 & d_{55} \end{pmatrix} h_{P_u}(\mathcal{A}) \in h_{\text{GL}(4|1)}(\mathcal{A}) / h_{P_u}(\mathcal{A})$$

with $\det C_1$ invertible. Then, by a change of coordinate (10) we can bring this matrix to the standard form detailed above

$$\pi \approx \begin{pmatrix} \mathbb{1}_2 & 0 & 0 \\ A & \mathbb{1}_2 & 0 \\ \alpha & 0 & 1 \end{pmatrix} h_{P_u}(\mathcal{A}), \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \alpha = (\alpha_1, \alpha_2),$$

with the entries of A in \mathcal{A}_0 and the entries of α in \mathcal{A}_1 . Its column vectors generate also the submodule $\langle a, b \rangle$.

The assumption that $\det C_1$ is invertible is equivalent to assume to be in the topological open set $|U_{12}| = |\text{Gr}| \cap |V_{12}|$, where V_{12} is the affine open set corresponding to the topological open set $|V_{12}|$ defined by taking in $\mathbf{P}(E)$ the coordinate q_{12} to be invertible. Consequently the coordinate superring of the affine open subvariety U_{12} of Gr corresponds to the projective localization of the Grassmannian superring in the coordinate q_{12} . In other words it consists of the elements of degree zero in

$$\mathbb{C}[q_{ij}q_{12}^{-1}, \lambda_j q_{12}^{-1}, a_{55}q_{12}^{-1}] \subset \mathcal{O}(\text{Gr})[q_{12}^{-1}].$$

As one can readily check, there are no relations among these generators so that the big cell U_{12} of Gr is the affine superspace with coordinate ring

$$\mathcal{O}(U_{12}) = \mathbb{C}[x_{ij}, \xi_j] \approx \mathbb{C}^{4|2}. \quad (11)$$

where we set $x_{ij} = q_{ij}q_{12}^{-1}$, $x_{55} = a_{55}q_{12}^{-1}$, $\xi_j = \lambda_j q_{12}^{-1}$.

We are now interested in the super subgroup of $\text{GL}(4|1)$ that preserves the big cell U_{12} . This is the lower parabolic sub-supergroup P_l (see [3]), whose functor of points is given in suitable coordinates as type

$$h_{P_l}(\mathcal{A}) = \left\{ \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix} \right\} \subset h_{\text{GL}(4|1)}(\mathcal{A})$$

where x and y are even, invertible 2×2 matrices, t is an even, arbitrary 2×2 matrix, η a 2×1 odd matrix, τ, ξ are 1×2 odd matrices and d is an invertible even element.

The action of the supergroup P_l on the big cell U_{12} is as follows,

$$h_{P_l}(\mathcal{A}) \times h_{U_{12}}(\mathcal{A}) \longrightarrow h_{U_{12}}(\mathcal{A})$$

$$\left(\begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix}, \begin{pmatrix} \mathbb{1}_2 \\ A \\ \alpha \end{pmatrix} \right) \longrightarrow \begin{pmatrix} \mathbb{1}_2 \\ A' \\ \alpha' \end{pmatrix},$$

where, using a transformation of $h_{P_u}(\mathcal{A})$ to revert the resulting matrix to the standard form (10), we have

$$\begin{pmatrix} \mathbb{1}_2 \\ A' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 \\ y(A + \eta\alpha)x^{-1} + t \\ d(\alpha + \tau + \xi A)x^{-1} \end{pmatrix}. \quad (12)$$

The subgroup with $\xi = 0$ is the super Poincaré group times dilations (compare with Eq. (14) in Ref [18]). In that case

$$d = \det x \det y.$$

6 Quantum chiral conformal superspace

In this section we give a quantum deformation of $\mathcal{O}(\text{Gr})$, discussed in the previous sections. This will yield a quantum deformation of the chiral conformal superspace together with the natural coaction of the conformal supergroup on it.

Definition 6.1. Let us define following Manin [23] the quantum matrix superalgebra.

$$M_q(m|n) =_{\text{def}} \mathbb{C}_q \langle a_{ij} \rangle / I_M$$

where $\mathbb{C}_q \langle a_{ij} \rangle$ denotes the free algebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the homogeneous variables a_{ij} and the ideal I_M is generated by the relations [23]:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \quad j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \quad i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \quad i < k, j > l \quad \text{or} \quad i > k, j < l$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} =$$

$$(-1)^{\pi(a_{ij})\pi(a_{kj})} (q^{-1} - q) a_{kj}a_{il} \quad i < k, j < l$$

where $p(i) = 0$ if $1 \leq i \leq m$, $p(i) = 1$ otherwise and $\pi(a_{ij}) = p(i) + p(j)$ denotes the parity of a_{ij} .

$M_q(m|n)$ is a bialgebra with the usual comultiplication and counit:

$$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}, \quad \mathcal{E}(a_{ij}) = \delta_{ij}.$$

We are ready to define the general linear supergroup which will be most interesting for us.

Definition 6.2. We define *quantum general linear supergroup*

$$\mathrm{GL}_q(m|n) =_{\mathrm{def}} M_q(m|n) \langle D_1^{-1}, D_2^{-1} \rangle$$

where D_1^{-1}, D_2^{-1} are even indeterminates such that:

$$D_1 D_1^{-1} = 1 = D_1^{-1} D_1, \quad D_2 D_2^{-1} = 1 = D_2^{-1} D_2$$

and

$$D_1 =_{\mathrm{def}} \sum_{\sigma \in S_m} (-q)^{-l(\sigma)} a_{1\sigma(1)} \cdots a_{m\sigma(m)}$$

$$D_2 =_{\mathrm{def}} \sum_{\sigma \in S_n} (-q)^{l(\sigma)} a_{m+1, m+\sigma(1)} \cdots a_{m+n, m+\sigma(n)}$$

are the quantum determinants of the diagonal blocks.

$\mathrm{GL}_q(m|n)$ is a Hopf algebra, where the comultiplication and counit are the same as in $M_q(m|n)$, while the antipode S is detailed in Ref. [14].

We now give the central definition in analogy with the ordinary setting (compare with Prop. 4.3).

Definition 6.3. Let the notation be as above. We define *quantum super Grassmannian* of $2|0$ planes in $4|1$ dimensional superspace as the non commutative superalgebra Gr_q generated by the following quantum super minors in $\mathrm{GL}_q(4|1)$:

$$D_{ij} = a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1}, \quad 1 \leq i < j \leq 4, \quad D_{55} = a_{51}a_{52}$$

$$D_{i5} = a_{i1}a_{52} - q^{-1}a_{i2}a_{51}, \quad 1 \leq i \leq 4.$$

For clarity let us write all the generators:

$$D_{12}, \quad D_{13}, \quad D_{14}, \quad D_{23}, \quad D_{24}, \quad D_{34}, \quad D_{55}, \quad D_{15}, \quad D_{25}, \quad D_{35}, \quad D_{45}$$

Notice that when $q = 1$ this is the coordinate ring of the super Grassmannian.

We need to work out the commutation relations and the quantum Plücker relations in order to be able to give a presentation of the quantum Grassmannian in terms of generators and relations.

Let us start with the commutation relations. With very similar calculations to the ones in Ref. [9] one finds the following relations:

- If i, j, k, l are *not* all distinct we have ($1 \leq i, j, k, l \leq 5$):

$$D_{ij}D_{kl} = q^{-1}D_{kl}D_{ij}, \quad (i, j) < (k, l)$$

where $<$ refers to the lexicographic ordering.

- If i, j, k, l are instead all distinct we have:

$$D_{ij}D_{kl} = q^{-2}D_{kl}D_{ij}, \quad 1 \leq i < j < k < l \leq 5$$

$$D_{ij}D_{kl} = q^{-2}D_{kl}D_{ij} - (q^{-1} - q)D_{ik}D_{jl}, \quad 1 \leq i < k < j < l \leq 5$$

$$D_{ij}D_{kl} = D_{kl}D_{ij}, \quad 1 \leq i < k < l < j \leq 5$$

- The only commutation relations that we are left to be shown are the following:

$$D_{ij}D_{55}, \quad D_{i5}D_{j5}, \quad D_{i5}D_{55}$$

After some computations one gets:

$$D_{ij}D_{55} = q^{-2}D_{55}D_{ij}, \quad 1 \leq i < j \leq 4$$

$$D_{i5}D_{j5} = -q^{-1}D_{j5}D_{i5} - (q^{-1} - q)D_{ij}D_{55} \quad 1 \leq i < j \leq 4$$

$$D_{i5}D_{55} = D_{55}D_{i5} = 0, \quad 1 \leq i \leq 4.$$

This concludes the discussion of the commutation relations. As for the Plücker relations, using the result for the non super setting (refer to [9]) we have

$$D_{12}D_{34} - q^{-1}D_{13}D_{24} + q^{-2}D_{14}D_{23} = 0$$

$$D_{ij}D_{k5} - q^{-1}D_{ik}D_{j5} + q^{-2}D_{i5}D_{jk} = 0, \quad 1 \leq i < j < k \leq 4$$

To this we must add the relations, which can be computed directly:

$$D_{i5}D_{j5} = qD_{ij}D_{55}, \quad 1 \leq i < j \leq 4.$$

The next proposition summarizes all of our calculations and the proof can be found in Ref. [3].

Proposition 6.4.

- *The quantum Grassmannian ring is given in terms of generators and relations as:*

$$Gr_q = \mathbb{C}_q \langle X_{ij} \rangle / I_{Gr}$$

where I_{Gr} is the two-sided ideal generated by the commutations and Plücker relations in the indeterminates X_{ij} . Moreover $Gr_q/(q-1) \cong \mathcal{O}(Gr)$ (see Section 3).

- *The quantum Grassmannian ring is the free ring over \mathbb{C}_q generated by the monomials in the quantum determinants:*

$$D_{i_1 j_1}, \dots, D_{i_r j_r}$$

where $(i_1, j_1), \dots, (i_r, j_r)$ form a semistandard tableau (for its definition refer to [3]).

The quantum Grassmannian that we have constructed admits a coaction of the quantum supergroup $GL_q(4|1)$. The proof of the following proposition amounts to a direct check (we refer again to Ref. [3] for more details).

Proposition 6.5. *Gr_q is a quantum homogeneous superspace for the quantum supergroup $GL_q(4|1)$, i. e., we have a coaction given via the restriction of the comultiplication of $GL_q(4|1)$:*

$$\Delta|_{Gr_q} : Gr_q \longrightarrow GL_q(4|1) \otimes Gr_q.$$

7 Quantum Minkowski superspace

We now turn to the quantum deformation of the big cell inside Gr_q ; it will be our model for the quantum Minkowski superspace.

In Section 5 we wrote the action of the lower parabolic supergroup P_l using the functor of points (12). We want now to translate it into the coaction language in order to make the generalization to the quantum setting.

Let $\mathcal{O}(P_l)$ be the superalgebra:

$$\mathcal{O}(P_l) := \mathcal{O}(\mathrm{GL}(4|1))/\mathcal{I}$$

where \mathcal{I} is the (two-sided) ideal generated by

$$g_{1j}, g_{2j}, \quad \text{for } j = 3, 4 \quad \text{and} \quad \gamma_{15}, \gamma_{25}.$$

This is the Hopf superalgebra coordinate superring of the lower parabolic subgroup P_l , with comultiplication naturally inherited by $\mathcal{O}(\mathrm{GL}(4|1))$.

In matrix form, for \mathcal{A} local, we have

$$h_{P_l}(\mathcal{A}) = \left\{ \begin{pmatrix} g_{11} & g_{12} & 0 & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} \right\} \subset h_{\mathrm{GL}(m|n)}(\mathcal{A}). \quad (13)$$

The superalgebra representing the big cell U_{12} can be realized as a subalgebra of $\mathcal{O}(P_l)$. In order to see this better, let us make the following two different changes of variables in P_l :

$$\begin{pmatrix} g_{11} & g_{12} & 0 & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ \tilde{\tau}x & d\xi & d \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix} \quad (14)$$

Notice that the only difference between the two sets of variables is that we replace τ with $\tilde{\tau}$ and we have:

$$d\tau = \tilde{\tau}x, \quad (15)$$

The next proposition tells us that these are sets of generators for $\mathcal{O}(P_l)$ and that having $\tilde{\tau}$ is essential to describe the big cell. Again for the proof we refer the reader to Ref. [3], while the explicit expressions for the generators come from a direct calculation.

Proposition 7.1.

1. *The Hopf superalgebra $\mathcal{O}(P_l)$ is generated by the following sets of variables:*

- $x, y, t, \tilde{\tau}, \xi, \eta$ and d ;
- x, y, t, τ, ξ, η and d

defined as

$$\begin{aligned}
x &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, & y &= \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}, \\
t &= \begin{pmatrix} -d_{23}d_{12}^{-1} & d_{13}d_{12}^{-1} \\ -d_{24}d_{12}^{-1} & d_{14}d_{12}^{-1} \end{pmatrix} & d &= g_{55} \\
\tilde{\tau} &= (-d_{25}d_{12}^{-1}, d_{15}d_{12}^{-1}) & \tau &= (g_{55}^{-1}\gamma_{51}, g_{55}^{-1}\gamma_{52}) \\
\eta &= \begin{pmatrix} d_{34}^{34-1}\gamma_{35} \\ d_{34}^{34-1}\gamma_{45} \end{pmatrix} & \xi &= (g_{55}^{-1}\gamma_{53}, g_{55}^{-1}\gamma_{54})
\end{aligned} \tag{16}$$

where for $1 \leq i < j \leq 4$

$$d_{ij} = g_{i1}g_{j2} - g_{j1}g_{i2}, \quad d_{i5} = g_{i1}\gamma_{52} - \gamma_{51}g_{i2}, \quad d_{34}^{34} = g_{33}g_{44} - g_{34}g_{43}.$$

2. *The subalgebra of $\mathcal{O}(P_l)$ generated by $(t, \tilde{\tau})$ coincides with the big cell superring $\mathcal{O}(U_{12})$ as defined in (11). It is given by the projective localization of $\mathcal{O}(\text{Gr})$ with respect to d_{12} .*

3. *There is a well defined coaction $\tilde{\Delta}$ of $\mathcal{O}(P_l)$ on $\mathcal{O}(U_{12})$ induced by the coproduct in $\mathcal{O}(P_l)$,*

$$\tilde{\Delta} : \mathcal{O}(U_{12}) \xrightarrow{\quad} \mathcal{O}(P_l) \otimes \mathcal{O}(U_{12})$$

which explicitly takes the form:

$$\tilde{\Delta}t_{ij} = t_{ij} \otimes 1 + y_{ia}S(x)_{bj} \otimes t_{ab} + y_i\eta_a S(x)_{bj} \otimes \tilde{\tau}_{jb},$$

$$\tilde{\Delta}\tilde{\tau}_j = (d \otimes 1)(\tau_a \otimes 1 + \xi_b \otimes t_{ba} + 1 \otimes \tilde{\tau}_a)(S(x)_{aj} \otimes 1),$$

The reader should notice right away that this is the dual to the expression (12).

We now turn to the quantum setting. In order to keep our notation minimal, we use the same letters as in the classical case to denote the generators of the quantum big cell and the quantum supergroups.

Let $\mathcal{O}(P_{l,q})$ be the superalgebra:

$$\mathcal{O}(P_{l,q}) := \mathcal{O}(\mathrm{GL}_q(4|1))/\mathcal{I}_q$$

where \mathcal{I}_q is the (two-sided) ideal in $\mathcal{O}(\mathrm{GL}_q(4|1))$ generated by

$$g_{1j}, g_{2j}, \quad \text{for } j = 3, 4 \quad \text{and} \quad \gamma_{15}, \gamma_{25}. \quad (17)$$

This is the Hopf superalgebra of the lower parabolic subgroup, again with comultiplication the one naturally inherited from $\mathcal{O}(\mathrm{GL}_q(4|1))$.

As in the classical case, it is convenient to change coordinates exactly in the same way (see 14), this time, however, paying extra attention to the order in which we take the variables. We can write the new coordinates for $\mathcal{O}(P_{l,q})$ explicitly:

$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad t = \begin{pmatrix} -q^{-1}D_{23}D_{12}^{-1} & D_{13}D_{12}^{-1} \\ -q^{-1}D_{24}D_{12}^{-1} & D_{14}D_{12}^{-1} \end{pmatrix}$$

$$y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}, \quad d = g_{55},$$

$$\tilde{\tau} = \begin{pmatrix} -q^{-1}D_{25}D_{12}^{-1} & D_{15}D_{12}^{-1} \end{pmatrix}, \quad \xi = \begin{pmatrix} g_{55}^{-1}\gamma_{53} & g_{55}^{-1}\gamma_{54} \end{pmatrix}$$

$$\eta = y^{-1} \begin{pmatrix} \gamma_{35} \\ \gamma_{45} \end{pmatrix} = (D_{34}^{34})^{-1} \begin{pmatrix} g_{44} & -q^{-1}g_{34} \\ -qg_{43} & g_{33} \end{pmatrix} = \begin{pmatrix} -q^{-1}D_{34}^{34-1}D_{34}^{45} \\ D_{34}^{34-1}D_{34}^{35} \end{pmatrix}$$

It is not hard to see that $\mathcal{O}(P_{l,q})$ is also generated by x, y, d, η, ξ and $\tilde{\tau}$.

Remark 7.2. The *quantum Poincaré supergroup times dilations* is the quotient of $\mathcal{O}(P_{l,q})$ by the ideal $\xi = 0$. In fact as one can readily check with a simple calculation, if $\mathcal{O}(Po)$ denotes the function algebra of the super (unquantized) Poincaré groups times dilations, we have that

$$(\mathcal{O}(P_{l,q}) / (\xi)) / (q - 1) \cong \mathcal{O}(Po).$$

One can also easily check that (ξ) is a Hopf ideal, so the comultiplication goes to the quotient. The quantum Poincaré supergroup times dilations is then generated by the images in the quotient of x, y, d, η and $\tilde{\tau}$. In matrix form, one has

$$\begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ \tilde{\tau}x & 0 & d \end{pmatrix}.$$

Explicitly in these coordinates its presentation is given as follows:

$$\mathcal{O}(P_{l,q}) / (\xi) = \mathbb{C}_q \langle t, x, y, \eta, \tau \rangle / I_{Po,q}$$

where $I_{Po,q}$ is the ideal generated by the following relations. The indeterminates x and y behave respectively as quantum (even) matrices, that is, their entries are subject to the relations 6.1. In other words we have for x (and similarly for y):

$$x_{11}x_{12} = q^{-1}x_{12}x_{11}, \quad x_{11}x_{21} = q^{-1}x_{21}x_{11}, \quad x_{21}x_{22} = q^{-1}x_{22}x_{21}$$

$$x_{12}x_{22} = q^{-1}x_{22}x_{12}, \quad x_{12}x_{21} = x_{21}x_{12}, \quad x_{11}x_{22} - x_{22}x_{11} = (q^{-1} - q)x_{12}x_{21}$$

Moreover the entries in x and y commute with each other. x and $t, \tilde{\tau}$ commute in the following way. Let $i = 1, 2, j = 3, 4$.

$$x_{1i}t_{j1} = q^{-1}t_{j1}x_{1i}, \quad x_{2i}t_{j1} = t_{j1}x_{2i},$$

$$x_{1i}\tilde{\tau}_{51} = q^{-1}\tilde{\tau}_{51}x_{1i}, \quad x_{2i}\tilde{\tau}_{51} = \tilde{\tau}_{51}x_{2i},$$

$$x_{1i}\tilde{\tau}_{52} = \tilde{\tau}_{52}x_{1i}, \quad x_{2i}\tilde{\tau}_{52} = q^{-1}\tilde{\tau}_{52}x_{2i}$$

x commutes with η and d . y, t and $\tilde{\tau}$ satisfy similar relations as x, t and τ that we leave to the reader as an exercise (the rows are exchanged with the

columns). y and η commute following the rules of quantum super matrices, very much the same calculation and relations expressed in 7.4. y and d commute. The commutation among t and $\tilde{\tau}$ are expressed in prop. 7.4. t and η commute. t , τ and d satisfy the following relations.

$$t_{ij}d = dt_{ij} - (q^{-1} - q)\eta_{i5}\tilde{\tau}_{5i}$$

$$\tilde{\tau}_{5j}d = d\tilde{\tau}_{5j}$$

$\tilde{\tau}$ and η commute with each other, while finally

$$\eta_{j5}d = q^{-1}d\eta_{j5}.$$

In analogy with the classical (non quantum) supersetting, we give the following definition.

Definition 7.3. We define the *quantum big cell* $\mathcal{O}_q(U_{12})$ as the subring of $\mathcal{O}(P_{lq})$ generated by t and $\tilde{\tau}$.

We compute now the quantum commutation relations among the generators of the quantum big cell $\mathcal{O}_q(U_{12})$, which is our chiral Minkowski superspace, and see that the quantum big cell admits a well defined coaction of the quantum supergroup $\mathcal{O}(P_q)$.

Proposition 7.4. *The quantum big cell superring $\mathcal{O}_q(U_{12})$ has the following presentation:*

$$\mathcal{O}_q(U_{12}) := \mathbb{C}_q\langle t_{ij}, \tilde{\tau}_{5j} \rangle / I_U, \quad 3 \leq i \leq 4, j = 1, 2$$

where I_U is the ideal generated by the relations:

$$t_{i1}t_{i2} = q t_{i2}t_{i1}, \quad t_{3j}t_{4j} = q^{-1} t_{4j}t_{3j}, \quad 1 \leq j \leq 2, \quad 3 \leq i \leq 4$$

$$t_{31}t_{42} = t_{42}t_{31}, \quad t_{32}t_{41} = t_{41}t_{32} + (q^{-1} - q)t_{42}t_{31},$$

$$\tilde{\tau}_{51}\tilde{\tau}_{52} = -q^{-1}\tilde{\tau}_{52}\tilde{\tau}_{51}, \quad t_{ij}\tilde{\tau}_{5j} = q^{-1}\tilde{\tau}_{5j}t_{ij}, \quad 1 \leq j \leq 2$$

$$t_{i1}\tilde{\tau}_{52} = \tilde{\tau}_{52}t_{i1}, \quad t_{i2}\tilde{\tau}_{51} = \tilde{\tau}_{51}t_{i2} + (q^{-1} - q)t_{i1}\tilde{\tau}_{52}.$$

As in the classical setting we have the following proposition.

Proposition 7.5. *The quantum big cell $\mathcal{O}_q(U_{12})$ admits a coaction of $\mathcal{O}(P_{l,q})$ obtained by restricting suitably the comultiplication in $\mathcal{O}(P_{l,q})$. In other words we have a well defined morphism:*

$$\tilde{\Delta} : \mathcal{O}_q(U_{12}) \longrightarrow \mathcal{O}(P_{l,q}) \otimes \mathcal{O}_q(U_{12})$$

satisfying the coaction properties and give explicitly by: (see 7.1),

$$\tilde{\Delta}t_{ij} = t_{ij} \otimes 1 + y_{ia}S(x)_{bj} \otimes t_{ab} + y_i\eta_aS(x)_{bj} \otimes \tilde{\tau}_{jb},$$

$$\tilde{\Delta}\tilde{\tau}_j = (d \otimes 1)(\tau_a \otimes 1 + \xi_b \otimes t_{ba} + 1 \otimes \tilde{\tau}_a)(S(x)_{aj} \otimes 1) \quad .$$

by choosing as before generators $x, y, t, d, \tau, \eta, \xi$ for $\mathcal{O}(P_{l,q})$ and $t, \tilde{\tau}$ for $\mathcal{O}_q(U_{12})$ with $d\tau = \tilde{\tau}x$.

Furthermore, this coaction goes down to a well defined coaction for the quantization of the super Poincaré group (see remark 7.2).

To compare with other deformations of the Minkowski space, we write here the even part of $\mathcal{O}_q(U_{12})$ in terms of the more familiar generators

$$t = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 + x^3 \end{pmatrix}.$$

The commutation relations of the generators x^μ are then [28]

$$\begin{aligned} x^0x^1 &= \frac{2}{q^{-1}+q}x^1x^0 + i\frac{q^{-1}-q}{q^{-1}+q}x^0x^2, \\ x^0x^2 &= \frac{2}{q^{-1}+q}x^2x^0 - i\frac{q^{-1}-q}{q^{-1}+q}x^0x^1, \\ x^0x^3 &= x^3x^0, \\ x^1x^2 &= \frac{i(q^{-1}+q)}{2} \left(-(x^0)^2 + (x^3)^2 + x^3x^0 - x^0x^3 \right), \\ x^1x^3 &= \frac{2}{q^{-1}+q}x^3x^1 - i\frac{q^{-1}-q}{q^{-1}+q}x^2x^3, \\ x^2x^3 &= \frac{2}{q^{-1}+q}x^3x^2 + i\frac{q^{-1}-q}{q^{-1}+q}x^1x^3. \end{aligned}$$

8 Chiral superfields in Minkowski superspace

In this section we wish to motivate the importance of the chiral conformal superspace and its quantum deformation in physics. We introduce chiral superfields in Minkowski superspace as they are used in physics. We start by introducing the complexified Minkowski space: the chiral superfields are a sub superalgebra of the coordinate superalgebra of Minkowski space. They can also be seen as the coordinate superalgebra of the chiral Minkowski superspace, which is complex.

8.1 Definitions

We consider the complexified Minkowski space \mathbb{C}^4 . The $N = 1$ *scalar superfields* on the complexified Minkowski space are elements of the commutative superalgebra

$$\mathcal{O}(\mathbb{C}^{4|4}) \equiv C^\infty(\mathbb{C}^4) \otimes \Lambda[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2], \quad (18)$$

where $\Lambda[\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2]$ is the Grassmann (or exterior) algebra generated by the odd variables $\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2$.

We will denote the coordinates (or generators) of the superspace as

$$\begin{aligned} x^\mu, & \quad \mu = 0, 1, 2, 3 \quad (\text{even coordinates}), \\ \theta^\alpha, \bar{\theta}^{\dot{\alpha}}, & \quad \alpha, \dot{\alpha} = 1, 2 \quad (\text{odd coordinates}), \end{aligned}$$

and a superfield, in terms of its *field components*, as

$$\begin{aligned} \Psi(x, \theta, \bar{\theta}) = & \psi_0(x) + \psi_\alpha(x)\theta^\alpha + \psi'_{\dot{\alpha}}(x)\bar{\theta}^{\dot{\alpha}} + \psi_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \psi_{\alpha\dot{\beta}}(x)\theta^\alpha\bar{\theta}^{\dot{\beta}} + \\ & \psi'_{\dot{\alpha}\dot{\beta}}(x)\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} + \psi_{\alpha\beta\dot{\gamma}}(x)\theta^\alpha\theta^\beta\bar{\theta}^{\dot{\gamma}} + \psi'_{\alpha\dot{\beta}\dot{\gamma}}(x)\theta^\alpha\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}} + \psi_{\alpha\beta\dot{\gamma}\dot{\delta}}(x)\theta^\alpha\theta^\beta\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}}. \end{aligned}$$

Action of the Lorentz group $\text{SO}(1,3)$. There is an action of the double covering of the complexified Lorentz group, $\text{Spin}(1,3)^c \approx \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ over $\mathbb{C}^{4|4}$. The even coordinates x^μ transform according to the fundamental representation of $\text{SO}(1,3)$ (V),

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu,$$

while θ and $\bar{\theta}$ are Weyl spinors (or half spinors). More precisely, the coordinates θ transform in one of the spinor representations, say $S^+ \approx (1/2, 0)$ and $\bar{\theta}$ transform in the opposite chirality representation, $S^- \approx (0, 1/2)$,

$$\theta^\alpha \mapsto S^\alpha{}_\beta \theta^\beta, \quad \bar{\theta}^{\dot{\alpha}} \mapsto \tilde{S}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}}.$$

The scalar superfields are invariant under the action of the Lorentz group,

$$\Psi(x, \theta, \bar{\theta}) = (R\Psi)(\Lambda^{-1}x, S^{-1}\theta, \tilde{S}^{-1}\bar{\theta}),$$

where $R\Psi$ is the superfield obtained by transforming the field components

$$R\psi_0(x) = \psi_0(x), \quad R\psi_\alpha(x) = S_\alpha{}^\beta \psi_\beta(x), \quad \dots$$

The hermitian matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

define a $\text{Spin}(1, 3)$ -morphism

$$\begin{aligned} S^+ \otimes S^- &\longrightarrow V \\ s^\alpha \otimes t^{\dot{\alpha}} &\longrightarrow s^\alpha \sigma_{\alpha\dot{\alpha}}^\mu t^{\dot{\alpha}}. \end{aligned}$$

Derivations. A *left derivation* of degree $m = 0, 1$ of a super algebra \mathcal{A} is a linear map $D^L : \mathcal{A} \mapsto \mathcal{A}$ such that

$$D^L(\Psi \cdot \Phi) = D^L(\Psi) \cdot \Phi + (-1)^{mp_\Psi} \Psi \cdot D^L(\Phi).$$

Graded left derivations span a \mathbb{Z}_2 -graded vector space (or *supervector space*).

In general, linear maps over a supervector space are also a \mathbb{Z}_2 -graded vector space. A map has degree 0 if it preserves the parity and degree 1 if it changes the parity. For the case of derivations of a commutative superalgebra, an even derivation has degree 0 as a linear map and an odd derivation has degree 1 as a linear map.

In the same way one defines *right derivations*,

$$D^R(\Psi \cdot \Phi) = (-1)^{mp_\Phi} D^R(\Psi) \cdot \Phi + \Psi \cdot D^R(\Phi).$$

Notice that derivations of degree zero are both, right and left derivations. Moreover, given a left derivation D^L of degree m one can define a right derivation D^R also of degree m in the following way

$$D^R\Psi = (-1)^{m(p_\Psi+1)} D^L\Psi. \tag{19}$$

Let us now focus on the commutative superalgebra $\mathcal{O}(\mathbb{C}^{4|4})$. We define the standard left derivations

$$\begin{aligned}\partial_\alpha^L \Psi &= \psi_\alpha + 2\psi_{\alpha\beta}\theta^\beta + \psi_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} + 2\psi_{\alpha\beta\dot{\gamma}}\theta^\beta\bar{\theta}^{\dot{\gamma}} + \psi'_{\alpha\dot{\beta}\dot{\gamma}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}} + 2\psi_{\alpha\beta\dot{\gamma}\dot{\delta}}\theta^\beta\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}}, \\ \partial_{\dot{\alpha}}^L \Psi &= \psi'_{\dot{\alpha}} - \psi_{\beta\dot{\alpha}}\theta^\beta + 2\psi'_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}} + \psi_{\gamma\beta\dot{\alpha}}\theta^\gamma\theta^\beta - 2\psi'_{\beta\dot{\alpha}\dot{\gamma}}\theta^\beta\bar{\theta}^{\dot{\gamma}} + 2\psi_{\gamma\beta\dot{\alpha}\dot{\delta}}\theta^\gamma\theta^\beta\bar{\theta}^{\dot{\delta}}.\end{aligned}$$

Also, using (19) one can define $\partial_\alpha^R, \partial_{\dot{\alpha}}^R$.

We consider now the odd left derivations

$$Q_\alpha^L = \partial_\alpha^L - i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{Q}_{\dot{\alpha}}^L = -\partial_{\dot{\alpha}}^L + i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu.$$

They satisfy the anticommutation rules

$$\{Q_\alpha^L, \bar{Q}_{\dot{\alpha}}^L\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, \quad \{Q_\alpha^L, Q_\beta^L\} = \{\bar{Q}_{\dot{\alpha}}^L, \bar{Q}_{\dot{\beta}}^L\} = 0,$$

with $\partial_\mu = \partial/\partial x^\mu$. Q^L and \bar{Q}^L are the *supersymmetry charges* or *supercharges*. Together with

$$P^\mu = -i\partial_\mu,$$

they form a Lie superalgebra, the *supertranslation algebra*, which then acts on the superspace $\mathbb{C}^{4|4}$.

Let us define another set of (left) derivations,

$$D_\alpha^L = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{D}_{\dot{\alpha}}^L = -\partial_{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu,$$

with anticommutation rules

$$\{D_\alpha^L, \bar{D}_{\dot{\alpha}}^L\} = -2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu, \quad \{D_\alpha^L, D_\beta^L\} = \{\bar{D}_{\dot{\alpha}}^L, \bar{D}_{\dot{\beta}}^L\} = 0.$$

They also form a Lie superalgebra, isomorphic to the supertranslation algebra. This can be seen by taking

$$Q^L \rightarrow -D^L, \quad \bar{Q}^L \rightarrow \bar{D}^L.$$

It is easy to see that the supercharges anticommute with the derivations D^L and \bar{D}^L . For this reason, D^L and \bar{D}^L are called *supersymmetric covariant derivatives* or simply *covariant derivatives*, although they are not related to any connection form.

We go now to the central definition.

Definition 8.1. A *chiral superfield* is a superfield Φ such that

$$\bar{D}_{\dot{\alpha}}^L \Phi = 0. \quad (20)$$

Because of the anticommuting properties of D 's and Q 's, we have that

$$\bar{D}_{\dot{\alpha}}^L \Phi = 0 \quad \Rightarrow \quad \bar{D}_{\dot{\alpha}}^L (Q_{\beta}^L \Phi) = 0, \quad \bar{D}_{\dot{\alpha}}^L (\bar{Q}_{\dot{\beta}}^L \Phi) = 0.$$

This means that the supertranslation algebra acts on the space of chiral superfields.

On the other hand, due to the derivation property,

$$\bar{D}_{\dot{\alpha}}^L (\Phi \Psi) = \bar{D}_{\dot{\alpha}}^L (\Phi) \Psi + (-1)^{p_{\Phi}} \Phi \bar{D}_{\dot{\alpha}}^L (\Psi),$$

we have that the product of two chiral superfields is again a chiral superfield.

8.2 Shifted coordinates

One can solve the constraint (20). Notice that the quantities

$$y^{\mu} = x^{\mu} + i\theta^{\alpha} \sigma_{\alpha\dot{\alpha}}^{\mu} \theta^{\dot{\alpha}}, \quad \theta^{\alpha} \quad (21)$$

satisfy

$$\bar{D}_{\dot{\alpha}}^L y^{\mu} = 0, \quad \bar{D}_{\dot{\alpha}}^L \theta^{\alpha} = 0.$$

Using the derivation property, any superfield of the form

$$\Phi(y^{\mu}, \theta), \quad \text{satisfies} \quad \bar{D}_{\dot{\alpha}}^L \Phi = 0$$

and so it is a chiral superfield. This is the general solution of (20).

We can make the change of coordinates

$$x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}} \longrightarrow y^{\mu} = x^{\mu} + i\theta^{\alpha} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\theta}^{\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}.$$

A superfield may be expressed in both coordinate systems

$$\Phi(x, \theta, \bar{\theta}) = \Phi'(y, \theta, \bar{\theta}).$$

The covariant derivatives and supersymmetry charges take the form

$$\begin{aligned} D_{\alpha}^L \Phi' &= \frac{\partial^L \Phi'}{\partial \theta^{\alpha}} + 2i\sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\theta}^{\dot{\alpha}} \frac{\partial^L \Phi'}{\partial y^{\mu}} & \bar{D}_{\dot{\alpha}}^L \Phi' &= -\frac{\partial^L \Phi'}{\partial \bar{\theta}^{\dot{\alpha}}}, \\ \bar{Q}_{\dot{\alpha}}^L \Phi' &= -\frac{\partial^L \Phi'}{\partial \bar{\theta}^{\dot{\alpha}}} + 2i\theta^{\alpha} \sigma_{\alpha\dot{\alpha}}^{\mu} \frac{\partial^L \Phi'}{\partial y^{\mu}} & Q_{\alpha}^L \Phi' &= \frac{\partial^L \Phi'}{\partial \theta^{\alpha}}. \end{aligned}$$

In the new coordinate system the chirality condition is simply

$$\frac{\partial^L \Phi'}{\partial \bar{\theta}^{\dot{\alpha}}} = 0,$$

so it is similar to a holomorphicity condition on the θ 's.

This shows that chiral scalar superfields are elements of the commutative superalgebra $\mathcal{O}(\mathbb{C}^{4|2}) = \mathbb{C}^\infty(\mathbb{C}^4) \otimes \Lambda[\theta^1, \theta^2]$. In the previous sections we realized this superspace as the big cell inside the chiral conformal superspace, which is the Grassmannian of 2|0-subspaces of $\mathbb{C}^{4|1}$.

The complete (non chiral) conformal superspace is in fact the flag space of 2|0-subspaces inside 2|1-subspaces of $\mathbb{C}^{4|1}$. On this supervariety one can put a reality condition, and the real Minkowski space is the big cell inside the flag. It is instructive to compare Eq. (21) with the incidence relation for the big cell of the flag manifold in Eq. (12) of Ref. [18]. We can then be convinced that the Grassmannian that we use to describe chiral superfields is inside the (complex) flag.

There are supersymmetric theories in physics (like Wess-Zumino models, or super Yang-Mills) that include in the formulation chiral superfields. In previous approaches it has been difficult to formulate them on non commutative superspaces (with non trivial commutation relations of the odd coordinates). The reason was that the covariant derivatives are not anymore derivations of the noncommutative superspace, and the chiral superfields do not form a superalgebra [10, 11]. Some proposals to solve these problems include the partial (explicit) breaking of supersymmetry [26, 11]. In our approach to quantization of superspace, the quantum chiral ring appears in a natural way, thus making possible the formulation of supersymmetric theories in non commutative superspaces. Also, the super variety and the supergroup acting on it become non commutative, the group law is not changed, so the physical symmetry principle remains intact. This is a virtue of the deformation based on quantum matrix groups.

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